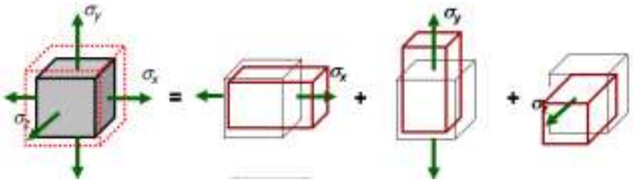
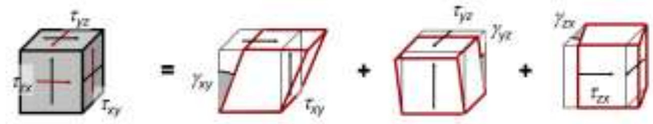
Finite element method (FEM1)

Lecture 3A. Two dimensional (2D) cases - CST element

03.2025

Hooke's law for an isotropic material in a three-dimensional state





$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

E – Young modulus
ν - *Poisson ratio*
G – shear modulus

$$G = \frac{E}{2(1+\nu)}$$

vector of stress components:

$$\underset{6\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

vector of strain components:

$$\underset{6\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

Constitutive matrix:

$$\underset{6\times 6}{[D]} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

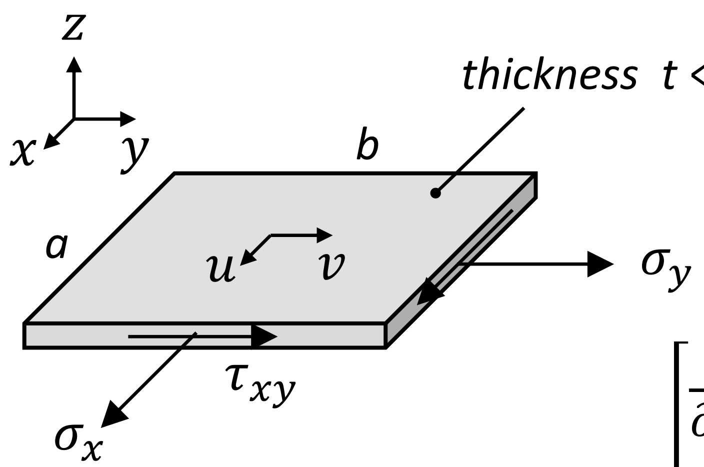
Plane stress (thin plates, shells)

σ_x ; σ_y ; $\sigma_z = 0$

τ_{xy} ; $\tau_{yz} = 0$; $\tau_{zx} = 0$

ε_x ; ε_y ; $\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$

γ_{xy} ; $\gamma_{yz} = 0$; $\gamma_{zx} = 0$



displacement vector:

$$\underset{1\times 2}{\lfloor u \rfloor} = \lfloor u, v \rfloor$$

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R]}\underset{2\times 1}{\{u\}}$$

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

Gradient matrix

vector of strain components:

$$\underset{1\times 3}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]}\,\underset{3\times 1}{\{\varepsilon\}}$$

Constitutive matrix for Plain Stress:

vector of stress components:

$$\underset{1\times 3}{\lfloor \sigma \rfloor} = \lfloor \sigma_x, \sigma_y, \tau_{xy} \rfloor$$

$$\underset{3\times 3}{[D]} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

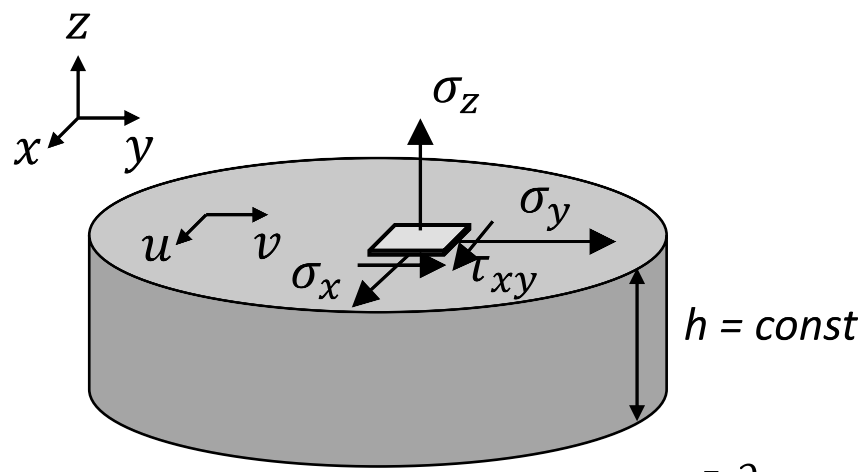
Plane strain (infinitely long pipe, prism and roller)

σ_x ; σ_y ; $\sigma_z = \nu(\sigma_x + \sigma_y)$

τ_{xy} ; $\tau_{yz} = 0$; $\tau_{zx} = 0$

ε_x ; ε_y ; $\varepsilon_z = 0$

γ_{xy} ; $\gamma_{yz} = 0$; $\gamma_{zx} = 0$



displacement vector:

$$\underset{1\times 2}{\lfloor u \rfloor} = \lfloor u, v \rfloor$$

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R]}\underset{2\times 1}{\{u\}}$$

Gradient matrix

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

vector of strain components:

$$\underset{1\times 3}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]}\ \underset{3\times 1}{\{\varepsilon\}}$$

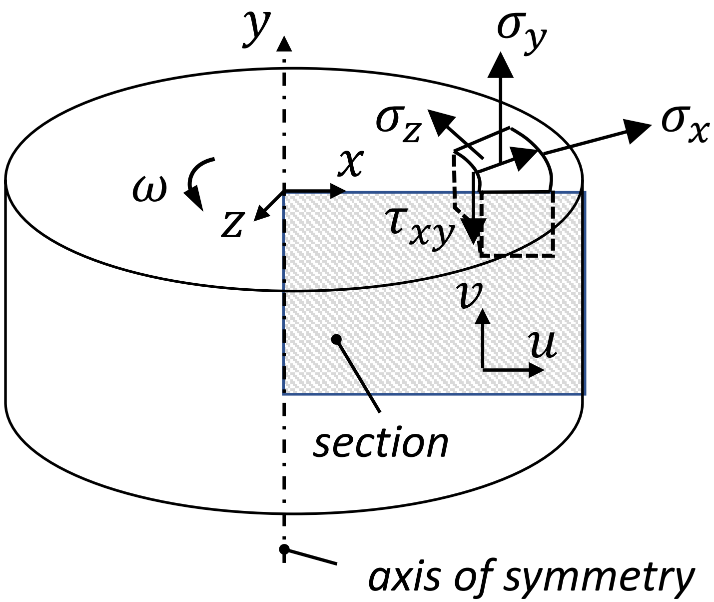
vector of stress components:

$$\underset{1\times 3}{\lfloor \sigma \rfloor} = \lfloor \sigma_x, \sigma_y, \tau_{xy} \rfloor$$

Constitutive matrix for plain strain:

$$\underset{3\times 3}{[D]} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

Axisymmetry (rotating disc)



$\sigma_x \quad ; \quad \sigma_y \quad ; \quad \sigma_z$

$\tau_{xy} \quad ; \quad \tau_{yz} = 0 \quad ; \quad \tau_{zx} = 0$

$\varepsilon_x \quad ; \quad \varepsilon_y \quad ; \quad \varepsilon_z = 0$

$\gamma_{xy} \quad ; \quad \gamma_{yz} = 0 \quad ; \quad \gamma_{zx} = 0$

directions:
x – radial
y – longitudinal
z – hoop

displacement vector: $\underset{1\times 2}{\lfloor u \rfloor} = \lfloor u, v \rfloor$

vector of strain components *:*

$$\underset{1\times 4}{\lfloor \varepsilon \rfloor} = \lfloor \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy} \rfloor$$

$$\underset{4\times 1}{\{\varepsilon\}} = \underset{4\times 2}{[R]}\underset{2\times 1}{\{u\}}$$

vector of stress components:

$$\underset{1\times 4}{\lfloor \sigma \rfloor} = \lfloor \sigma_x, \sigma_y, \sigma_z, \tau_{xy} \rfloor$$

$$\underset{4\times 1}{\{\sigma\}} = \underset{4\times 4}{[D]}\underset{4\times 1}{\{\varepsilon\}}$$

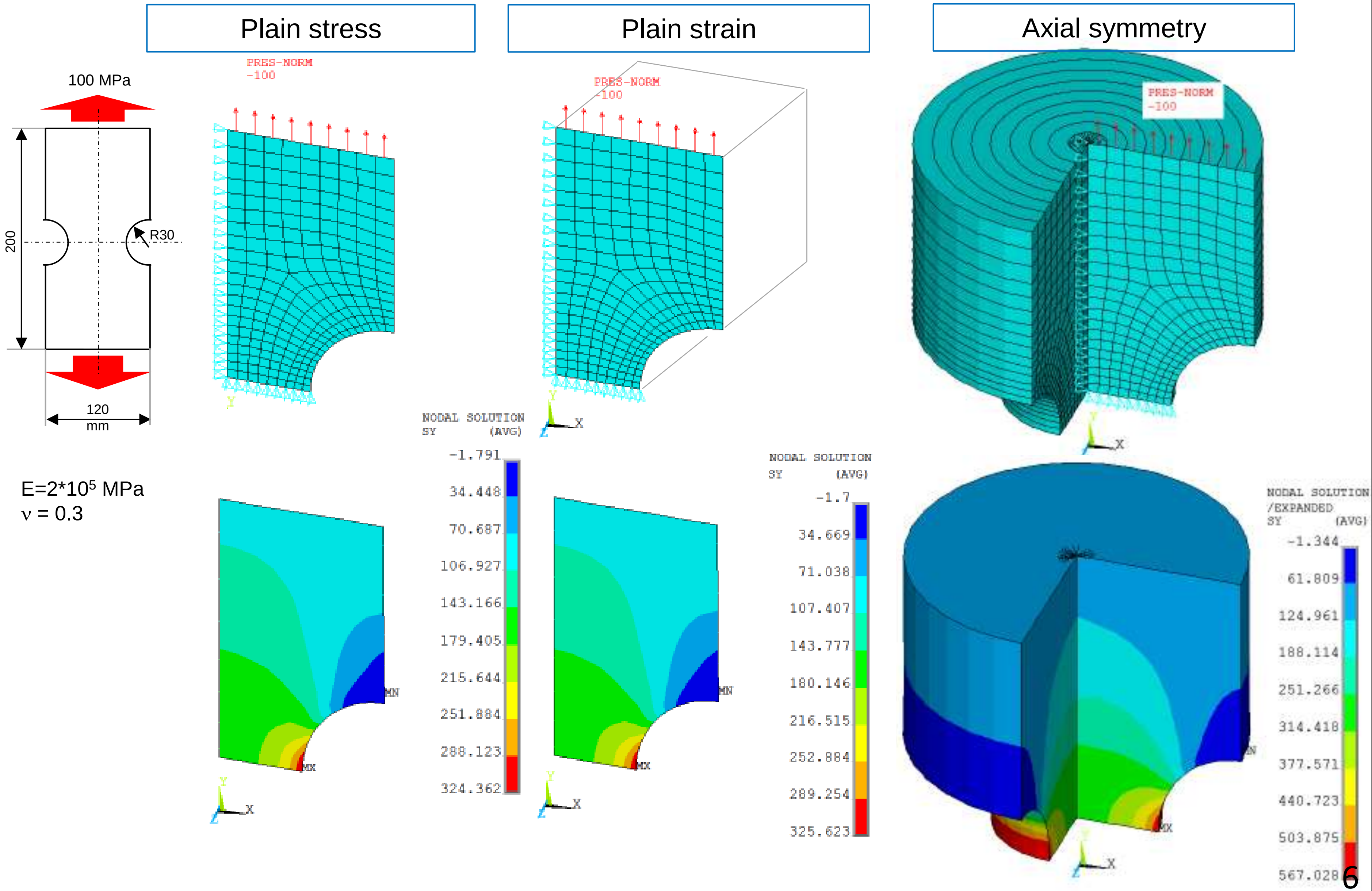
gradient matrix:

$$\underset{4\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ \frac{1}{x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

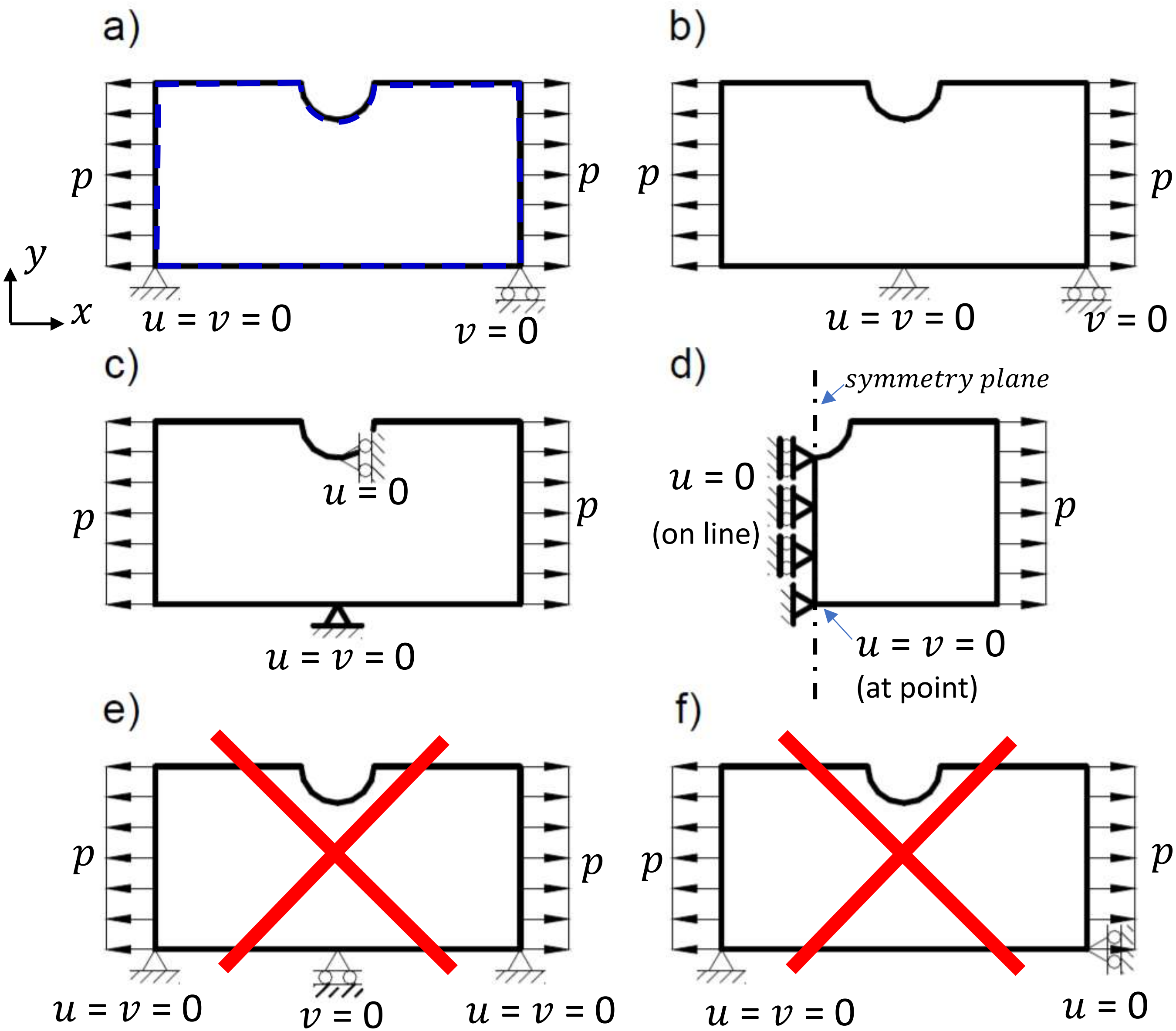
$$\underset{4\times 4}{[D]} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

Constitutive matrix for axial symmetry

Example of using the 2D element option



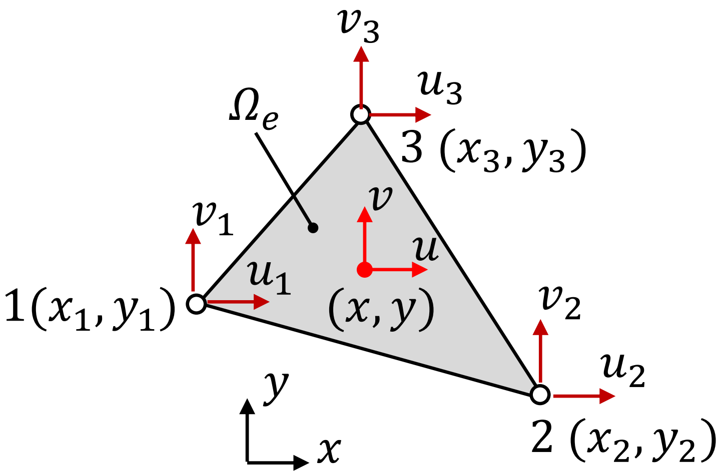
Support conditions for a 2D plate loaded with forces in equilibrium



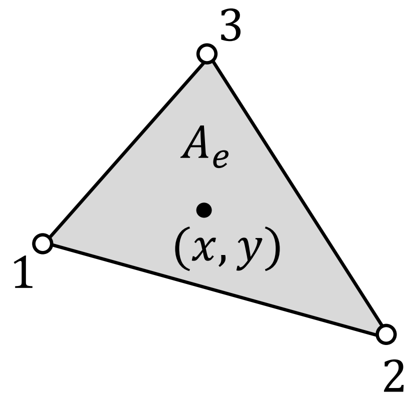
Even in the case of self-balanced loading, it is necessary to assign a number of degrees of freedom for the model that will prevent the possibility of movement as a rigid body.
At the same time, we should not restrict the freedom of deformation so that the stiffness matrix does not become singular and the solution ambiguous.

Correct constraints:
(constrained rigid body motion and correct deformation):
a, b, c, d

CST *(Constant Strain Triangle)* **finite element** (2D, 3-node triangle)



$n = 3$; $n_p = 2$ → $n_e = n \cdot n_p = 6$

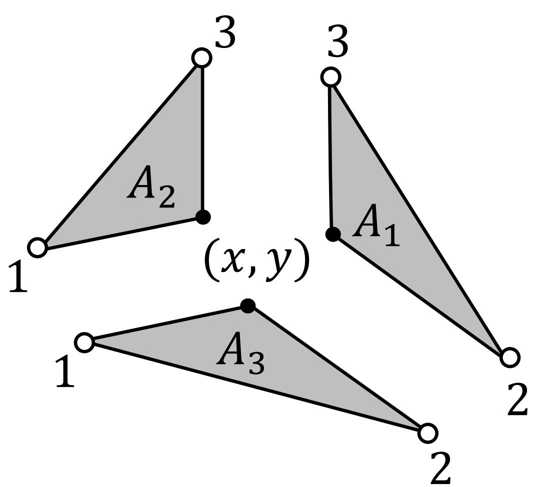


$$\int_{\Omega_e} d\Omega_e = A_e \cdot t_e$$

(A_e: area; t_e: thickness)

Area of a triangle with vertices 1,2,3:

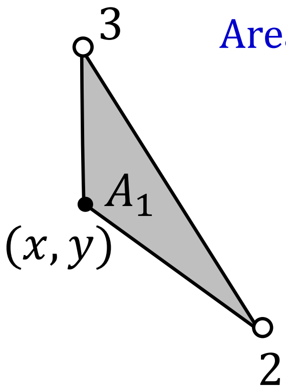
$$A_e = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1}{2}$$



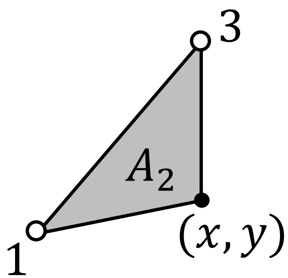
$$A_e = A_1(x, y) + A_2(x, y) + A_3(x, y)$$

CST finite element

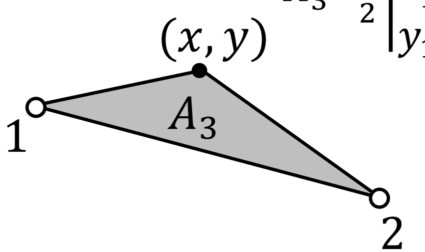
Area coordinates as functions of coordinates (x, y):



$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} \quad ; \quad \boldsymbol{L_1} = \frac{A_1}{A_e} = \frac{1}{2A}\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = \frac{\overbrace{x_2y_3 - x_3y_2}^{a_1} + \overbrace{(y_2 - y_3)}^{b_1}x + \overbrace{(x_3 - x_2)}^{c_1}y}{2A}$$



$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix} \quad ; \quad \boldsymbol{L_2} = \frac{A_2}{A_e} = \frac{1}{2A}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix} = \frac{\overbrace{x_3y_1 - x_1y_3}^{a_2} + \overbrace{(y_3 - y_1)}^{b_2}x + \overbrace{(x_1 - x_3)}^{c_2}y}{2A}$$



$$A_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} \quad ; \quad \boldsymbol{L_3} = \frac{A_3}{A_e} = \frac{1}{2A}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} = \frac{\overbrace{x_1y_2 - x_2y_1}^{a_3} + \overbrace{(y_1 - y_2)}^{b_3}x + \overbrace{(x_2 - x_1)}^{c_3}y}{2A}$$

$$A_e = A_1(x, y) + A_2(x, y) + A_3(x, y)$$

$$1 = L_1(x, y) + L_2(x, y) + L_3(x, y)$$

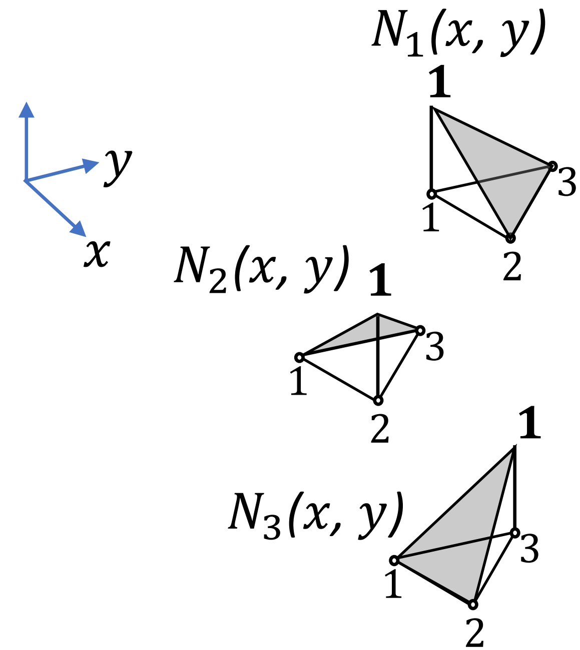
Shape functions of the CST element

shape functions =normalized area coordinates:

$$N_1(x,y) = L_1(x,y) = \frac{A_1(x,y)}{A_e} = \frac{1}{2A_e}(a_1 + b_1 x + c_1 y)$$

$$N_2(x,y) = L_2(x,y) = \frac{A_2(x,y)}{A_e} = \frac{1}{2A_e}(a_2 + b_2 x + c_2 y)$$

$$N_3(x,y) = L_3(x,y) = \frac{A_3(x,y)}{A_e} = \frac{1}{2A_e}(a_3 + b_3 x + c_3 y)$$



$$N_1(x,y) + N_2(x,y) + N_3(x,y) = 1$$

node	$N_1(x,y)$	$N_2(x,y)$	$N_3(x,y)$
1	**1**	0	0
2	0	**1**	0
3	0	0	**1**

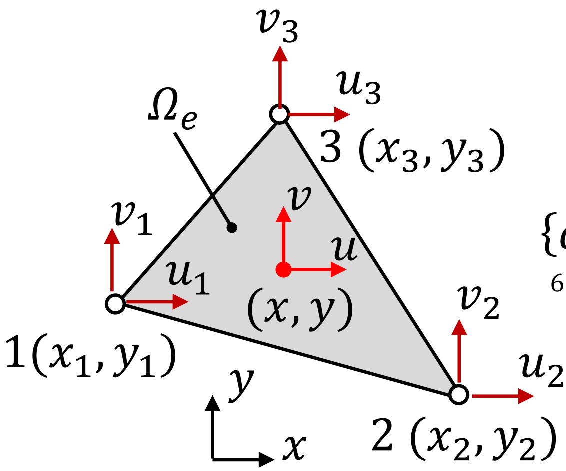
where:

$a_1 = x_2 y_3 - x_3 y_2$; $a_2 = x_3 y_1 - x_1 y_3$; $a_3 = x_1 y_2 - x_2 y_1$

$b_1 = y_2 - y_3$; $b_2 = y_3 - y_1$; $b_3 = y_1 - y_2$

$c_1 = x_3 - x_2$; $c_2 = x_1 - x_3$; $c_3 = x_2 - x_1$

Isoparametric mapping in the CST element



$$\underset{6\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_e$$

vector of shape functions:

$$\underset{1\times 3}{\lfloor N(x,y) \rfloor} = \lfloor N_1(x,y), N_2(x,y), N_3(x,y) \rfloor$$

vectors of nodal coordinates:

$$\underset{3\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad ; \quad \underset{3\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

coordinates of any point are based on shape functions and nodal parameters:

$$x = \underset{1\times 3}{\lfloor N(x,y) \rfloor} \underset{3\times 1}{\{x_i\}_e} = N_1(x,y)x_1 + N_2(x,y)x_2 + N_3(x,y)x_3$$

$$y = \underset{1\times 3}{\lfloor N(x,y) \rfloor} \underset{3\times 1}{\{y_i\}_e} = N_1(x,y)y_1 + N_2(x,y)y_2 + N_3(x,y)y_3$$

displacements at any point:

$$\underset{2\times 1}{\{u(x,y)\}} = \underset{2\times 6}{[N(x,y)]} \underset{6\times 1}{\{q\}_e}$$

Isoparametric mapping- the same shape functions used for geometry and displacements

Strain-displacement matrix of the CST element

strain vector for plane stress or plane strain conditions:

$$\underset{3\times1}{\{\varepsilon\}} = \underset{3\times2}{[R]}\,\underset{2\times1}{\{u\}} = \underset{3\times2}{[R]}\,\underset{2\times6}{[N]}\underset{6\times1}{\{q\}_e} =$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \underset{6\times1}{\{q\}_e} =$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \underset{6\times1}{\{q\}_e} = \underset{3\times6}{[B]}\,\underset{6\times1}{\{q\}_e}$$

$$[B] = \frac{1}{2A_e}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \rightarrow \underset{3\times1}{\{\varepsilon\}} = \underset{3\times6}{[B]}\,\underset{6\times1}{\{q\}_e}$$ – strain is constant

$$\underset{3\times1}{\{\sigma\}} = \underset{3\times3}{[D]}\,\underset{3\times1}{\{\varepsilon\}}$$ – stress is constant

CST – Constant Strain Triangle

Elastic strain energy in the CST element. Local stiffness matrix

elastic strain energy in a finite element:

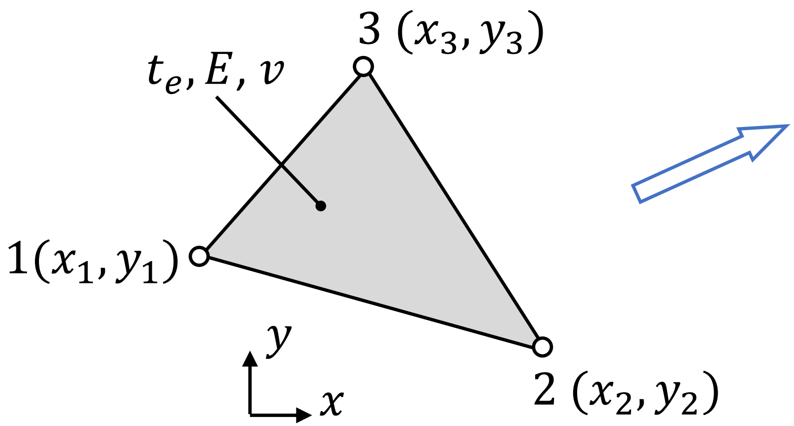
$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}} \int_{\Omega_e} d\Omega_e = \frac{1}{2} \underset{1\times 6}{\lfloor q \rfloor_e} \underset{6\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 6}{[B]} \underset{6\times 1}{\{q\}_e} A_e t_e =$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]} \underset{3\times 1}{\{\varepsilon\}}$$

$$= \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e$$

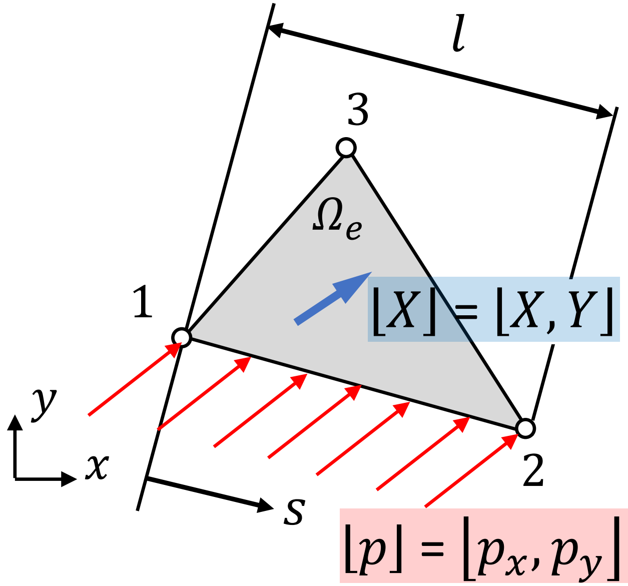
$$\underset{1\times 3}{\lfloor \varepsilon \rfloor} = \underset{1\times 6}{\lfloor q \rfloor_e} \underset{6\times 3}{[B]^T} \qquad \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 6}{[B]} \underset{6\times 1}{\{q\}_e}$$

local stiffness matrix:



$$\underset{6\times 6}{[k]_e} = A_e t_e \underset{6\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 6}{[B]}$$

Potential energy of loading in the CST element



potential energy of loading in a finite element:

$$W_e = \int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 1}{\{u\}}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 1}{\{u\}}\, d\Gamma_{pe} =$$

$$\underset{2\times 1}{\{u\}} = \underset{2\times 6}{[N]} \underset{6\times 1}{\{q\}_e}$$

$$= \int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]} \underset{6\times 1}{\{q\}_e}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]} \underset{6\times 1}{\{q\}_e}\, d\Gamma_{pe} =$$

$$= \left(\int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]}\, d\Gamma_{pe}\right) \underset{6\times 1}{\{q\}_e} = \left(\underset{1\times 6}{\lfloor F^X \rfloor_e} + \underset{1\times 6}{\lfloor F^p \rfloor_e}\right)\underset{6\times 1}{\{q\}_e} = \underset{1\times 6}{\lfloor F \rfloor_e} \underset{6\times 1}{\{q\}_e}$$

equivalent load vector due to mass forces:

$$\underset{1\times 6}{\lfloor F^X \rfloor_e} = t_e \int_{A_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]} dA_e$$

;

equivalent load vector due to surface load:

$$\underset{1\times 6}{\lfloor F^p \rfloor_e} = t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]}\, ds$$

Components of equivalent load vector in the CST element

$$\underset{1\times 6}{\lfloor F^X \rfloor_e} = t_e \int_{A_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 6}{[N]} dA_e \quad ; \quad \underset{1\times 6}{\lfloor F^p \rfloor_e} = t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 6}{[N]}\, ds$$

equivalent load vector due to mass forces:

$$\underset{1\times 6}{\lfloor F^X \rfloor_e} = t_e \int_{A_e} \lfloor X, Y \rfloor \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} dA_e =$$

$$= t_e \int_{A_e} \lfloor XN_1, YN_1, XN_2, YN_2, XN_3, YN_3 \rfloor dA_e = \left\lfloor F_{1e}^X, F_{2e}^X, F_{3e}^X, F_{4e}^X, F_{5e}^X, F_{6e}^X \right\rfloor$$

equivalent load vector due to surface load:

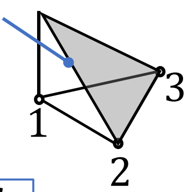
$$\underset{1\times 6}{\lfloor F^p \rfloor_e} = t_e \int_0^l \lfloor p_x, p_y \rfloor \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} ds =$$

$$= t_e \int_0^l \lfloor p_x, p_y \rfloor \begin{bmatrix} 1-\frac{s}{l} & 0 & \frac{s}{l} & 0 & 0 & 0 \\ 0 & 1-\frac{s}{l} & 0 & \frac{s}{l} & 0 & 0 \end{bmatrix} ds =$$

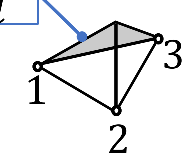
$$= t_e \int_0^l \left\lfloor p_x\left(1-\frac{s}{l}\right), p_y\left(1-\frac{s}{l}\right), p_x \frac{s}{l}, p_y \frac{s}{l}, 0, 0 \right\rfloor ds =$$

$$= \left\lfloor F_{1e}^p, F_{2e}^p, F_{3e}^p, F_{4e}^p, F_{5e}^p, F_{6e}^p \right\rfloor$$

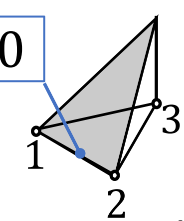
$N_1(s)|_{1-2} = 1 - \frac{s}{l}$



$N_2(s)|_{1-2} = \frac{s}{l}$



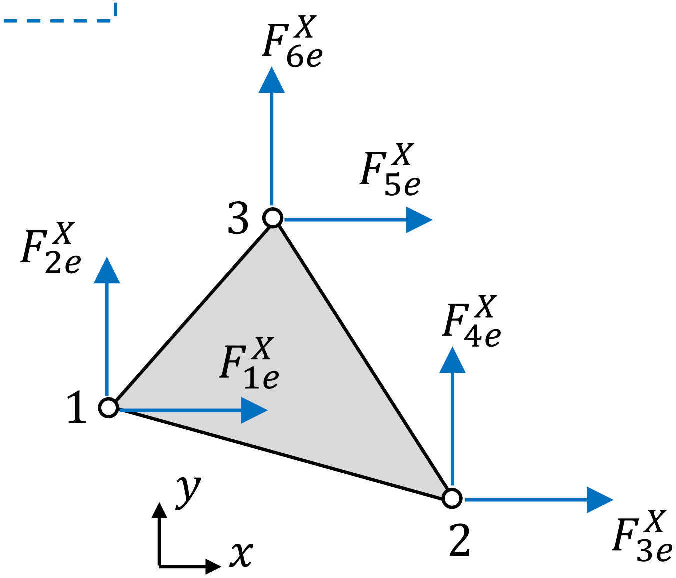
$N_3(x,y)|_{1-2} = 0$



Equivalent load vector in the CST element

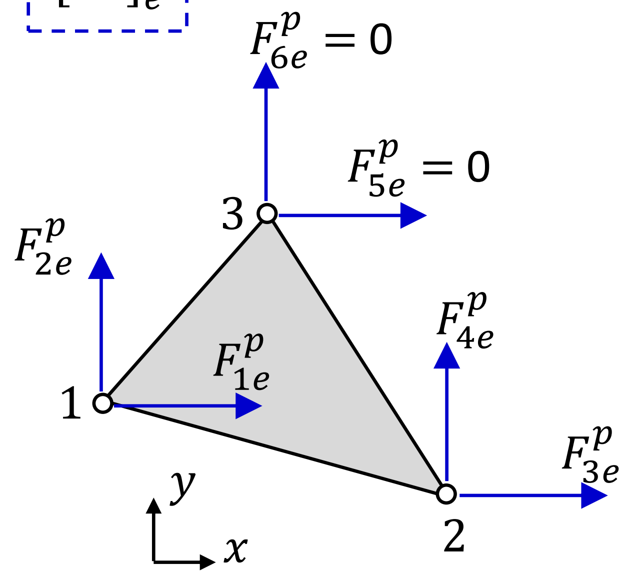
equivalent load vector
due to mass forces:

$\lfloor F^X \rfloor_e$



equivalent load vector
due to surface load:

$\lfloor F^p \rfloor_e$

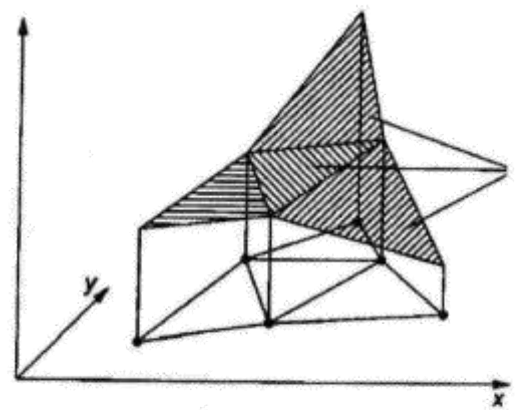


equivalent load vector:

$$\lfloor F \rfloor_e = \lfloor\, F_{1e}^X + F_{1e}^p \,,\; F_{2e}^X + F_{2e}^p \,,\; F_{3e}^X + F_{3e}^p \,,\; F_{4e}^X + F_{4e}^p \,,\; F_{5e}^X + F_{5e}^p \,,\; F_{6e}^X + F_{6e}^p \rfloor$$

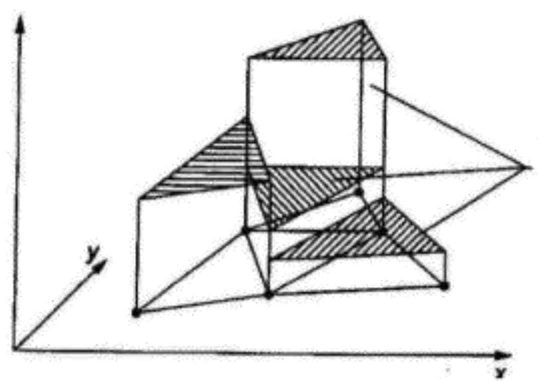
Results in the CST element

DOF solution : $u(\mathrm{x}, \mathrm{y}), v(\mathrm{x}, \mathrm{y})$



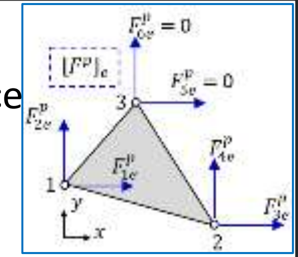
linear functions of coordinates (x, y)

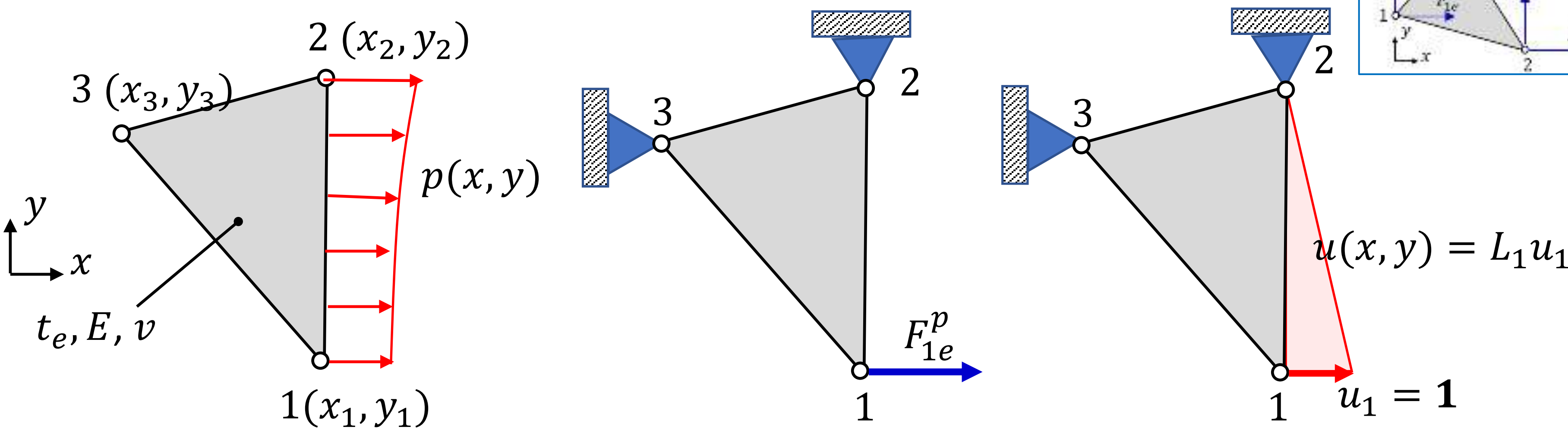
element solution: $\underset{3\times 1}{\{\sigma\}}, \underset{3\times 1}{\{\varepsilon\}}$



constant

Example 1: Determination of the equivalent force in the CST element due to surface force





equivalent load vector *due to* surface load:

$$\lfloor F^p \rfloor_e = t_e \int_0^l \lfloor p \rfloor [N]\, ds$$

The work of the equivalent force F_{1e}^p on displacement 1

The work of load $p(x,y)$ on displacement $u(x,y)$

$$F_{1e}^p \cdot 1 = t_e \int_0^l p(x,y)\, u(x,y)\, dy$$

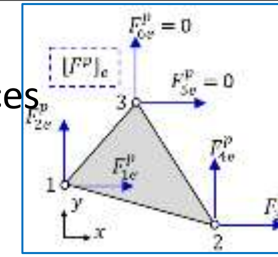
$$F_{1e}^p = t_e \int_0^l p(x,y)\, L_1\, dy$$

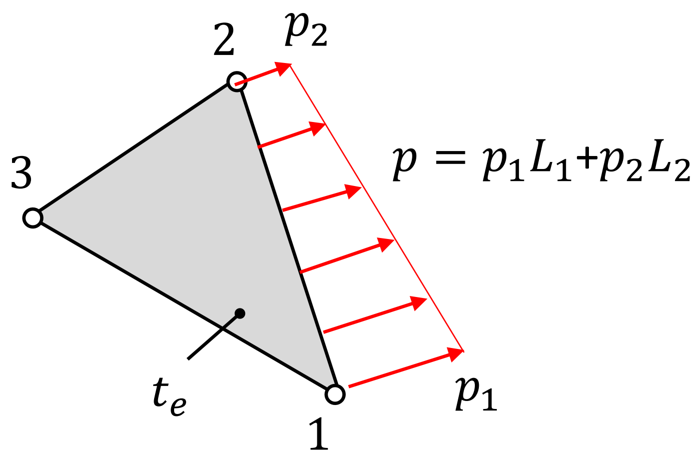
Integrals of barycentric functions:

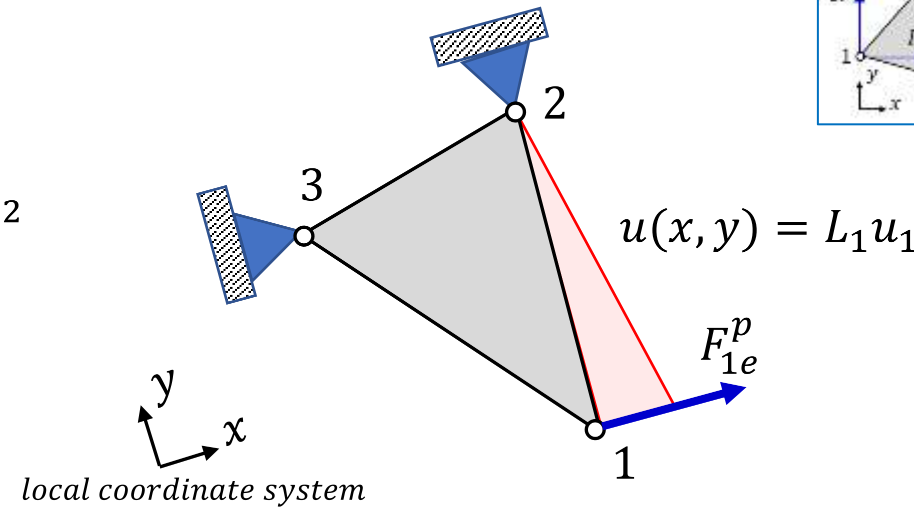
$$J_1 = \int_0^l L_1^q L_2^r\, dl = \frac{q!r!\, l}{(q+r+1)!}$$

$$J_2 = \int_{A_e} L_1^q L_2^r\, L_3^t dA_e = \frac{q!r!t!}{(q+r+t+2)!} 2A_e$$

Example 2: Determination of the equivalent force in the CST element due to surface forces







The work of the equivalent force F_{1e}^p on displacement 1

The work of load $p(x,y)$ on displacement $u(x,y)$

$$F_{1e}^p \cdot 1 = t_e \int_0^l p(x,y)\, u(x,y)\, dy \;\longrightarrow\; F_{1e}^p = t_e \int_0^l (p_1L_1+p_2L_2)\, L_1\, dy$$

$$F_{1e}^p = t_e\left(p_1 \int_0^l L_1^2\, dy + p_2 \int_0^l L_1L_2\, dy\right) = t_e\left(p_1 \frac{2!0!\, l}{(2+0+1)!} + p_2 \frac{1!1!\, l}{(1+1+1)!}\right) = t_e\left(\tfrac{1}{3}p_1 l + \tfrac{1}{6}p_2 l\right)$$

Integral of barycentric function:

$$J_1 = \int_0^l L_1^q L_2^r\, dl = \frac{q!r!\, l}{(q+r+1)!}$$

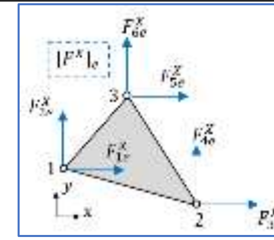
For $p_1 = p_2$:

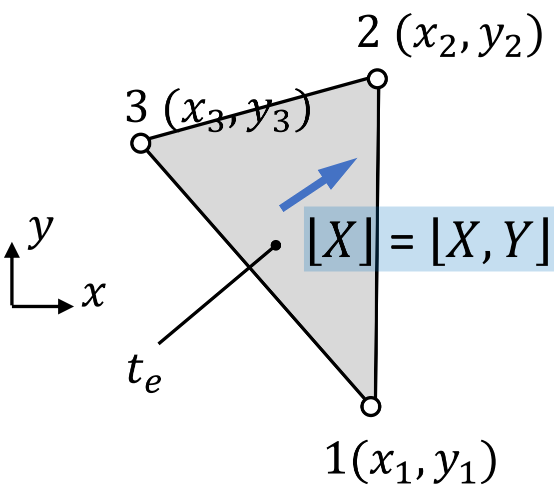
$$F_{1e}^p = t_e \int_0^l p\, L_1\, dy = t_e p \frac{1!0!\, l}{(1+0+1)!} = \frac{p\, l}{2} t_e$$

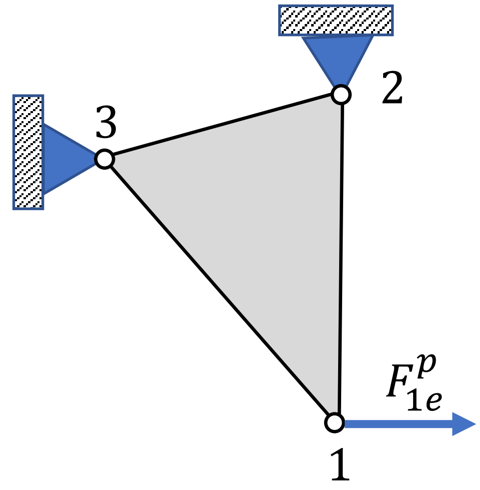
For $p_2 = 0$:

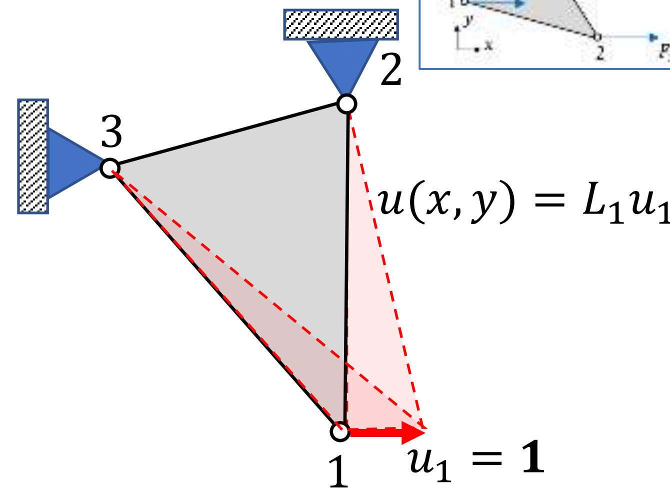
$$F_{1e}^p = t_e \int_0^l p_1 L_1^2\, dy = t_e p \frac{2!0!\, l}{(2+0+1)!} = \frac{p\, l}{3} t_e$$

Example 3: Determination of the equivalent force in the CST element due to mass load









equivalent load vector *due to mass* load :

$$\lfloor F^X \rfloor_e = t_e \int_{A_e} \lfloor X \rfloor [N] dA_e$$

The work of the equivalent force F_{1e}^X on displacement 1 — *The work of load X* on displacement $u(x, y)$

$$F_{1e}^X \cdot 1 = t_e \int_{A_e} X\, u(x, y)\, dA_e$$

$$F_{2e}^X \cdot 1 = t_e \int_{A_e} Y\, v(x, y)\, dA_e$$

$$F_{1e}^X = t_e \int_{A_e} X\, L_1\, dA_e$$

$$F_{2e}^X = t_e \int_{A_e} Y\, L_1\, dA_e$$

Integral of barycentric function :

$$J_2 = \int_{A_e} L_1^q L_2^r\, L_3^t dA_e = \frac{q!r!t!}{(q+r+t+2)!} 2A_e$$

For $X = const\ i\ Y = 0$:

$$F_{1e}^X = t_e X \int_{A_e} L_1\, dA_e = t_e X \frac{1!0!0!}{(1+0+0+2)!} 2A_e = X \frac{A_e t_e}{3}$$

$$F_{2e}^X = 0$$